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SOME BOUNDS ON SUM CONNECTIVITY AND PRODUCT CONNECTIVITY ZAGREB-K-BANHATTI INDICES OF GRAPHS

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ABSTRACT

The connectivity indices are applied to measure the chemical characteristics of compound in Chemical Graph Theory. In this paper, we introduce the sum connectivity Zagreb-*K*-Banhatti index and product connectivity Zagreb-K-Banhatti index of a graph. We provide lower and upper bounds for the sum connectivity Zagreb-*K*-Banhatti index and product connectivity Zagreb-*K*-Banhatti index of a graph in terms of Zagreb and *K*-Banhatti indices.

Keywords: *Graph, sum connectivity Zagreb-K-Banhatti index, product connectivity Zagreb-K-Banhatti index.*
Mathematics Subject Classification: *05C05, 05C12, 05C35.*

1. INTRODUCTION

Let G be a simple, connected graph with n vertices and m edges with vertex set $V(G)$ and edge set $E(G)$. The degree $d_G(u)$ of a vertex u is the number of vertices adjacent to u. If $e=uv$ is an edge of G, then the vertex u and edge e are incident as are v and e. The vertices and edges of a graph are called its elements. The degree of an edge $e=uv$ in G is defined by $d_G(e) = d_G(u) + d_G(v) – 2$. For all further notation and terminology, we refer the reader to [1].

A molecular graph is a graph such that its vertices correspond to the atoms and the edges to the bounds. Chemical Graph Theory is a branch of Mathematical Chemistry which has an important effect on the development of the Chemical Sciences. A topological index is a numeric quantity from the structure of a molecule. There are numerous molecular descriptors, which are also referred to as topological indices, that have found some applications in Theoretical Chemistry, especially in QSPR/QSAR study, see [2, 3, 4].

The first and second Zagreb indices take into account the contributions of pairs of adjacent vertices. These indices were introduced by Gutman et al. in [5], defined as

$$M_1(G) = \sum_{u\in V(G)} d_G(u)^2 = \sum_{uv\in E(G)} \left[d_G(u)+d_G(v)\right]$$

$$M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v).$$

These indices have been extensively studied in [6, 7, 8].

The modified second Zagreb index [9] is defined as

$$M_2^*(G) = \sum_{uv\in E(G)} \frac{1}{d_G(u)d_G(v)}.$$

The sum connectivity index [10] of a graph G is defined as

$$S(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_G(u)+d_G(v)}}.$$

The product connectivity index [11] of a graph G is defined as

$$P(G)=\sum_{uv\in E(G)}\frac{1}{\sqrt{d_G(u)d_G(v)}}.$$

The general product connectivity index [12] of a graph G is defined as

$$P_\alpha(G)=\sum_{uv\in E(G)}\left[d_G(u)d_G(v)\right]^\alpha$$

where α is a real number.

More details on these types of connectivity indices, we refer to [13, 14, 15].

In [16], Miličević et al. introduced the first and second reformulated Zagreb indices of a graph G in terms of edge degrees instead of vertex degrees and defined as

$$EM_1(G)=\sum_{e\in E(G)}d_G(e)^2, \qquad EM_2(G)=\sum_{e\sim f}d_G(e)d_G(f).$$

where $e \sim f$ means that the edges e and f are adjacent.

We define the sum connectivity reformulated index of a graph G as

$$SEM(G)=\sum_{e\sim f}\frac{1}{\sqrt{d_G(e)+d_G(f)}}.$$

We also define the product connectivity reformulated index of a graph G as

$$PEM(G)=\sum_{e\sim f}\frac{1}{\sqrt{d_G(e)d_G(f)}}.$$

The reformulated Zagreb indices were studied, for example, in [17, 18, 19].

The first and second K-Banhatti indices take into account the contributions of pairs of incident elements. The first and second K-Banhatti indices were introduced by Kulli in [20], defined as

$$B_1(G)=\sum_{ue}\left[d_G(u)+d_G(e)\right], \qquad B_2(G)=\sum_{ue}d_G(u)d_G(e),$$

where ue means that the vertex u and edge e are incident.

In [21], Kulli et al. introduced the sum connectivity Banhatti index of a graph G, which is defined as

$$SB(G)=\sum_{ue}\frac{1}{\sqrt{d_G(u)+d_G(e)}}.$$

The produce connectivity Banhatti index was introduced by Kulli et al. in [22] and defined it as

$$PB(G)=\sum_{ue}\frac{1}{\sqrt{d_G(u)d_G(e)}}.$$

The K-Banhatti indices have been studied extensively. For their applications and mathematical properties, see [23, 24, 25, 26, 27].

Motivated by the work on the Zagreb and K-Banhatti indices, Kulli et al. introduced the Zagreb-K-Banhatti index [28] of a graph G and defined it as

$$MB(G)=\sum_{\substack{a \text{ is either adjacent}\\ \text{or incident to } b}}\left[d_G(a)+d_G(b)\right]$$

where a and b are elements of G.

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The second Zagreb-*K*-Banhatti index [29] of a graph *G* is defined as

$$MB_2(G) = \sum_{\substack{a \text{ is either adjacent} \\ \text{or incident to } b}} d_G(a) d_G(b)$$

The first and second hyper Zagreb-*K*-Banhatti indices were introduced and studied by Kulli in [30].

Based on the successful consideration of Zagreb-*K*-Banhatti indices, we introduce the sum connectivity Zagreb-*K*-Banhatti index and product connectivity Zagreb-*K*-Banhatti index of a graph *G* and they are defined as

$$SMB(G) = \sum_{\substack{a \text{ is either adjacent} \\ \text{or incident to } b}} \frac{1}{\sqrt{d_G(a) + d_G(b)}},$$

$$PMB(G) = \sum_{\substack{a \text{ is either adjacent} \\ \text{or incident to } b}} \frac{1}{\sqrt{d_G(a) d_G(b)}}.$$

In this study, we obtain some lower and upper bounds for *SMB*(*G*) and *PMB*(*G*) in terms of some degree based topological indices.

2. BOUNDS ON SUM CONNECTIVITY ZAGREB-*K*-BANHATTI, ZAGREB, *K*-BANHATTI- TYPE INDICES

Theorem 1. Let *G* be a graph with $n \geq 3$ vertices and *m* edges. Then

$$SMB(G) = S(G) + SEM(G) + SB(G).$$

Proof: Let *G* be a graph with $n \geq 3$ vertices and *m* edges. Then

$$SMB(G) = \sum_{\substack{a \text{ is either adjacent} \\ \text{or incident to } b}} \frac{1}{\sqrt{d_G(a) + d_G(b)}}$$

$$= \sum_{ab \in E(G)} \frac{1}{\sqrt{d_G(a) + d_G(b)}} + \sum_{e,f \in E(G), e \sim f} \frac{1}{\sqrt{d_G(a) + d_G(b)}} + \sum_{a(ab)} \frac{1}{\sqrt{d_G(a) + d_G(b)}}$$

$$= S(G) + SEM(G) + SB(G).$$

We use the following inequality to prove our next result.

Theorem 2 [23]. For any (*n*, *m*)-connected graph *G* with $\delta(G) \geq 2$ and $n \geq 3$ vertices,

$$SB(G) \leq 2S(G).$$

Theorem 3. For any (*n*, *m*)-connected graph *G* with $n \geq 3$ vertices and $\delta(G) \geq 2$,

$$SMB(G) \leq 3S(G) + SEM(G).$$

Proof: From Theorem 1, we have

$$SMB(G) = S(G) + SEM(G) + SB(G).$$

Using Theorem 2, we obtain

$$SMB(G) \leq 3S(G) + SEM(G).$$

We use the following result to prove our next result.

Theorem 4 [31]. Let *G* be a graph with *n* vertices and *m* edges. Then

$$S(G) \leq \sqrt{\frac{mP(G)}{2}},$$

with equality if and only if *G* is regular.

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Theorem 5. For any (n, m)-connected graph G with $n \leq 3$ vertices,

$$SMB(G) \leq \frac{3}{\sqrt{2}}\sqrt{mP(G)} + SEM(G).$$

Proof: From Theorem 3, we have

$$SMB(G) \leq 3S(G) + SEM(G).$$

Using Theorem 4, we obtain

$$SMB(G) \leq \frac{3}{\sqrt{2}}\sqrt{mP(G)} + SEM(G).$$

Theorem 6. For any (n, m) connected graph G with $n \leq 3$ *vertices and* $m \geq 1$ edges,

$$SMB(G) \leq \frac{3}{2}\sqrt{mn} + SEM(G).$$

Proof: In [31], $S(G) \leq \frac{\sqrt{mn}}{2}$ with equality if and only if G is regular.

Using this inequality and Theorem 3, we get,

$$SMB(G) \leq \frac{3}{2}\sqrt{mn} + SEM(G).$$

We use the following result to establish our next result.

Theorem 7 [10]. Let G be a graph with n vertices and maximum degree $\Delta(G)$. Then

$$S(G) \leq \frac{n\sqrt{\Delta(G)}}{2\sqrt{2}},$$

with equality if and only if G is regular of degree $\Delta(G)$.

Theorem 8. Let G be a graph with $n \geq 3$ vertices, m edges and maximum degree $\Delta(G)$. Then

$$SMB(G) \leq \frac{3n\sqrt{\Delta(G)}}{2\sqrt{2}} + SEM(G).$$

Further, $SMB(G) \leq \frac{3n\sqrt{n-1}}{2\sqrt{2}} + SEM(G).$

Proof: From Theorem 3, we have

$$SMB(G) \leq 3S(G) + SEM(G).$$

Using Theorem 7, we obtain

$$SMB(G) \leq \frac{3n\sqrt{\Delta(G)}}{2\sqrt{2}} + SEM(G).$$

Since $\Delta(G) \leq n-1$, we get

$$SMB(G) \leq \frac{3n\sqrt{n-1}}{2\sqrt{2}} + SEM(G).$$

We use the following result to prove our next results.

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Theorem 9 [23]. For any (n, m)-connected graph G with $\delta(G)\geq 2$ and $n\geq 3$ vertices,

(1) $$SB(G)\leq\sqrt{m(m+1)P(G)}.$$

(2) $$SB(G)\leq\sqrt{mM_1(G)}.$$

(3) $$SB(G)\leq\sqrt{m(m+1)M_2^*(G)}.$$

Theorem 10. For any (n, m)-connected graph G with $\delta(G) \geq 2$ and $n \geq 3$ vertices,

$$SMB(G)\leq\sqrt{mP(G)}\left(\frac{1}{\sqrt{2}}+\sqrt{m+1}\right)+SEM(G).$$

Proof: From Theorem 1, we have

$$SMB(G)=S(G)+SEM(G)+SB(G).$$

Using Theorem 4, we get

$$SMB(G)\leq\sqrt{\frac{mP(G)}{2}}+SEM(G)+SB(G).$$

Using Theorem 9(1), we obtain

$$SMB(G)\leq\sqrt{\frac{mP(G)}{2}}+SEM(G)+\sqrt{m(m+1)P(G)}.$$

Thus $$SMB(G)\leq\sqrt{mP(G)}\left(\frac{1}{\sqrt{2}}+\sqrt{m+1}\right)+SEM(G).$$

Theorem 11. For any (n, m)-connected graph G with $\delta(G) \geq 2$ and $n \geq 3$ vertices,

$$SMB(G)\leq\frac{n\sqrt{\Delta(G)}}{2\sqrt{2}}+SEM(G)+\sqrt{mM_1(G)}.$$

Proof: From Theorem 1, we have

$$SMB(G)=S(G)+SEM(G)+SB(G).$$

Using Theorem 7, we get

$$SMB(G)\leq\frac{n\sqrt{\Delta(G)}}{2\sqrt{2}}+SEM(G)+SB(G).$$

Using Theorem 9(2), we obtain

$$SMB(G)\leq\frac{n\sqrt{\Delta(G)}}{2\sqrt{2}}+SEM(G)+\sqrt{mM_1(G)}.$$

Theorem 12. For any (n, m)-connected graph with $\delta(G) \geq 2$ and $n \geq 3$ vertices,

$$SMB(G)\leq\frac{\sqrt{nm}}{2}+\sqrt{m(m+1)M_2^*(G)}+SEM(G).$$

Proof: From Theorem 1, we have

$$SMB(G)=S(G)+SEM(G)+SB(G).$$

Since $S(G)\leq\frac{\sqrt{nm}}{2}$, and Theorem 9(3), we obtain

$$SMB(G)\leq\frac{\sqrt{nm}}{2}+\sqrt{m(m+1)M_2^*(G)}+SEM(G).$$

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We use the following results to prove our next result.

Theorem 13 [10]. Let G be a graph with $m \geq 1$ edges. Then

$$S(G) \leq \frac{m\sqrt{m}}{\sqrt{M_1(G)}}.$$

with equality if and only if $d_G(u)+ d_G(v)$ is constant for every edge uv of G.

Theorem 14 [23]. For any (n, m)-connected graph G with $n \geq 3$ vertices,

$$SB(G) \leq \frac{(2m)^{\frac{3}{2}}}{\sqrt{B_1(G)}}.$$

with equality if and only if G is regular.

Theorem 15. For any (n, m)-connected graph with G with $n \geq 3$ vertices,

$$SMB(G) \geq m\sqrt{m}\left(\frac{1}{\sqrt{M_1(G)}} + \frac{2\sqrt{2}}{\sqrt{B_1(G)}}\right) + SEM(G).$$

Proof: From Theorem 1, we have

$$SMB(G) = S(G) + SEM(G) + SB(G).$$

From Theorems 13 and 14, we obtain

$$SMB(G) \geq \frac{m\sqrt{m}}{\sqrt{M_1(G)}} + SEM(G) + \frac{(2m)^{\frac{3}{2}}}{\sqrt{B_1(G)}}.$$

Therefore $SMB(G) \geq m\sqrt{m}\left(\frac{1}{\sqrt{M_1(G)}} + \frac{2\sqrt{2}}{\sqrt{B_1(G)}}\right) + SEM(G).$

We use the following result to establish our next result.

Theorem 16 [10]. Let G be a graph with $n \geq 5$ vertices containing no isolated vertices. Then

$$S(G) \geq \frac{n-1}{\sqrt{n}}$$

with equality if and only if G is a star S_n.

Theorem 17. For any (n, m)-connected graph with G with $n \geq 5$ vertices,

$$SMB(G) \geq \frac{n-1}{\sqrt{n}} + \frac{(2m)^{\frac{3}{2}}}{\sqrt{B_1(G)}} + SEM(G).$$

Proof: From Theorem 1, we have

$$SMB(G) = S(G) + SEM(G) + SB(G).$$

Using Theorems 14 and 16, we obtain

$$SMB(G) \geq \frac{n-1}{\sqrt{n}} + \frac{(2m)^{\frac{3}{2}}}{\sqrt{B_1(G)}} + SEM(G).$$

We use the following result to obtain our next result.

Theorem 18 [23]. For any (n, m) connected graph G with $\delta(G) \geq 2$ and $n \geq 3$ vertices,

$$\frac{n\sqrt{2}}{\sqrt{(n-1)(n-2)}} \leq SB(G) \leq \sqrt{mn}.$$

Further, equality holds in lower bound if and only if $G = C_3$; and equality holds in upper bound if and only if $G = C_n$; $n \geq 3$.

Theorem 19. For any (n, m)-connected graph with G with $\delta(G) \geq 2$ and $n \geq 3$ vertices,

$$\frac{n\sqrt{2}}{\sqrt{(n-1)(n-2)}} + S(G) + SEM(G) \leq SMB(G) \leq \sqrt{mn} + S(G) + SEM(G).$$

Proof: From Theorem 1, we have

$$SMB(G) = S(G) + SEM(G) + SB(G).$$

Using Theorem 18, we obtain

$$\frac{n\sqrt{2}}{\sqrt{(n-1)(n-2)}} + S(G) + SEM(G) \leq SMB(G) \leq \sqrt{mn} + S(G) + SEM(G).$$

3. BOUNDS ON PRODUCT CONNECTIVITY ZAGREB-*K*-BANHATTI, ZAGREB, *K*-BANHATTI-TYPE INDICES

Theorem 20. Let G be a graph with $n \geq 3$ vertices and m edges. Then

$$PMB(G) = P(G) + PEM(G) + PB(G).$$

Proof: Let G be a graph with $n \geq 3$ vertices and m edges. Then

$$\begin{aligned} PMB(G) &= \sum_{\substack{a \text{ is either adjacent} \\ \text{or incident to } b}} \frac{1}{\sqrt{d_G(a)d_G(b)}} \\ &= \sum_{ab \in E(G)} \frac{1}{\sqrt{d_G(a)d_G(b)}} + \sum_{e,f \in E(G), e \sim f} \frac{1}{\sqrt{d_G(e)d_G(f)}} + \sum_{a(ab)} \frac{1}{\sqrt{d_G(a)d_G(b)}} \\ &= P(G) + PEM(G) + PB(G). \end{aligned}$$

We use the following result to prove our next result.

Theorem 21 [22]. For any connected graph G with $n \geq 3$ vertices,

$$PB(G) > P(G).$$

Theorem 22. For any (n, m)-connected graph G with $n \geq 3$ vertices,

$$PMB(G) > 2P(G) + PEM(G).$$

Proof: From Theorem 20, we have

$$PMB(G) = P(G) + PEM(G) + PB(G).$$

Using Theorem 21, we obtain

$$PMB(G) > 2P(G) + PEM(G).$$

We use the following result to establish our next result.

Theorem 23[32]. Let G be a graph with n vertices and with minimum degree δ(G) and maximum degree Δ(G). Then

$$P(G) \geq \frac{\delta(G)\Delta(G)}{\delta(G)+\Delta(G)} n$$

Equality holds only when G is (δ(G), Δ(G)) biregular.

Theorem 24. For any (n, m)-connected graph G with $n \geq 3$ vertices, minimum degree δ(G) and maximum degree Δ(G),

$$PMB(G) > \frac{2n\sqrt{\delta(G)\Delta(G)}}{\delta(G)+\Delta(G)} + PEM(G).$$

Proof: From Theorem 22, we have

$$PMB(G) > 2P(G) + PEM(G).$$

$$\geq \frac{2n\sqrt{\delta(G)\Delta(G)}}{\delta(G)+\Delta(G)} + PEM(G), \text{ by Theorem 23.}$$

We use the following result to establish our next result.

Theorem 25 [22]. For any (n, m)-connected graph G with $n \geq 3$ vertices,

$$PB(G) \geq \frac{n\delta(G)}{\sqrt{2\Delta(G)(\Delta(G)-1)}}.$$

Equality holds if and only if G is regular.

Theorem 26. For any (n, m)-connected graph G with $n \geq 3$ vertices, minimum degree δ(G) and minimum degree Δ(G),

$$PMB(G) \geq n\left[\frac{\sqrt{\delta(G)\Delta(G)}}{\delta(G)+\Delta(G)} + \frac{\delta(G)}{\sqrt{2\Delta(G)(\Delta(G)-1)}}\right] + PEM(G).$$

Proof: From Theorem 20, we have

$$PMB(G) = P(G) + PEM(G) + PB(G).$$

Using Theorem 23, we obtain

$$PMB(G) \geq \frac{n\sqrt{\delta(G)\Delta(G)}}{\delta(G)+\Delta(G)} + PB(G) + PEM(G).$$

Again by using Theorem 25, we get

$$PMB(G) \geq \frac{n\sqrt{\delta(G)\Delta(G)}}{\delta(G)+\Delta(G)} + \frac{n\delta(G)}{\sqrt{2\Delta(G)(\Delta(G)-1)}} + PEM(G).$$

Thus, $$PMB(G) \geq n\left[\frac{\sqrt{\delta(G)\Delta(G)}}{\delta(G)+\Delta(G)} + \frac{\delta(G)}{\sqrt{2\Delta(G)(\Delta(G)-1)}}\right] + PEM(G).$$

We use the following result to establish our next result.

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Theorem 27 [13]. Let G be a graph with n vertices and with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$. Then

$$P_\alpha(G) \geq \frac{n\Delta(G)^\alpha \, \delta(G)^{1+\alpha}}{2}.$$

with equality if and only if G is regular.

Therefore $$P(G) = P_{-\frac{1}{2}}(G) \geq \frac{n}{2}\sqrt{\frac{\delta(G)}{\Delta(G)}}. \qquad (1)$$

Theorem 28. For any (n, m)-connected graph with $n \geq 3$ vertices, and with minimum degree $\delta(G)$, maximum degree $\Delta(G)$,

$$PMB(G) \geq n\left(\frac{1}{2}\sqrt{\frac{\delta(G)}{\Delta(G)}} + \frac{\delta(G)}{\sqrt{2\Delta(G)(\Delta(G)-1)}}\right) + PEM(G).$$

Proof: From Theorem 20, we have

$$PMB(G) = P(G) + PEM(G) + PB(G)$$

Using inequality (1), we get

$$PMB(G) \geq \frac{n}{2}\sqrt{\frac{\delta(G)}{\Delta(G)}} + PEM(G) + PB(G).$$

Using Theorem 25, we obtain

$$PMB(G) \geq n\left(\frac{1}{2}\sqrt{\frac{\delta(G)}{\Delta(G)}} + \frac{\delta(G)}{\sqrt{2\Delta(G)(\Delta(G)-1)}}\right) + PEM(G).$$

Theorem 29. For any (n, m)-connected graph G with $n\geq 3$ vertices and with minimum degree $\delta(G)$, maximum degree $\Delta(G)$,

$$PMB(G) > n\left(\frac{\delta(G)}{\Delta(G)}\right)^{\frac{1}{2}} + PEM(G).$$

Proof: From Theorem 22, we have

$$PMB(G) > 2P(G) + PEM(G)$$

Using inequality (1), we obtain

$$PMB(G) > n\left(\frac{\delta(G)}{\Delta(G)}\right)^{\frac{1}{2}} + PEM(G).$$

We use the following result to establish our next result.

Theorem 30 [22]. For any (n, m)-connected graph G with $\delta(G) \geq 2$ and $n \geq 3$ vertices,

$$PB(G) \leq SB(G).$$

Further equality is attained if and only if $G = C_n$.

Theorem 31. For any (n, m)-connected graph G with $\delta(G) \geq 2$ and $n \geq 3$ vertices,

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$$PMB(G) \le P(G) + \sqrt{m(m+1)P(G)} + PEM(G).$$

Proof: From Theorem 20, we have

$$PMB(G) = P(G) + PEM(G) + PB(G).$$

Using Theorem 30, we get

$$PMB(G) \le P(G) + SB(G) + PEM(G).$$

Using Theorem 9(1), we obtain

$$PMB(G) \le P(G) + \sqrt{m(m+1)P(G)} + PEM(G).$$

We use the following result to prove our next result.

Theorem 32 [13]. Let G be a graph with n vertices and minimum degree $\delta(G)$. Let $\alpha \in \left(-\infty, -\frac{1}{2}\right]$. Then

$$P_\alpha(G) \le \frac{n\delta(G)^{1+2\alpha}}{2}$$

Therefore $$P(G) = P_{-\frac{1}{2}}(G) \le \frac{n}{2} \qquad (2)$$

Theorem 33. For any (n, m)-connected graph G with $n \ge 3$ vertices and $\delta(G) \ge 2$,

$$PMB(G) \le \frac{n}{2} + \sqrt{mM_1(G)} + PEM(G).$$

Proof: From Theorem 20, we have

$$PMB(G) = P(G) + PB(G) + PEM(G).$$

Using inequality (2) and Theorem 30, we get

$$PMB(G) \le \frac{n}{2} + SB(G) + PEM(G). \qquad (3)$$

Using Theorem 9(2), we obtain

$$PMB(G) \le \frac{n}{2} + \sqrt{mM_1(G)} + PEM(G).$$

Theorem 34. For any (n, m)-connected graph G with $\delta(G) \ge 2$ and $n \ge 3$ vertices,

$$PMB(G) \le \frac{n}{2} + \sqrt{m(m+1)M_2^*(G)} + PEM(G).$$

Proof: From inequality (3), we have

$$PMB(G) \le \frac{n}{2} + SB(G) + SEM(G).$$

Using Theorem 9(3), we obtain

$$PMB(G) \le \frac{n}{2} + \sqrt{m(m+1)M_2^*(G)} + PEM(G).$$

We use the following result to establish our next result.

Theorem 35 [22]. For any connected graph G with $\delta(G) \ge 2$ and $n \ge 3$ vertices,

$$\sqrt{\frac{2\delta(G)}{\Delta(G)-1}}P(G) \le PB(G) \le \sqrt{\frac{2\Delta(G)}{\delta(G)-1}}P(G).$$

Theorem 36. For any connected graph G with $\delta(G) \ge 2$ and $n \ge 3$ vertices,

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$$\left(1+\sqrt{\frac{2\delta(G)}{\Delta(G)-1}}\right)P(G)+PEM(G)\leq PMB(G)\leq\left(1+\sqrt{\frac{2\Delta(G)}{\delta(G)-1}}\right)P(G)+PEM(G).$$

Proof: From Theorem 20, we have

$$PMB(G)=P(G)+PB(G)+PEM(G).$$

Using Theorem 35, we obtain

$$\left(1+\sqrt{\frac{2\delta(G)}{\Delta(G)-1}}\right)P(G)+PEM(G)\leq PMB(G)\leq\left(1+\sqrt{\frac{2\Delta(G)}{\delta(G)-1}}\right)P(G)+PEM(G).$$

Theorem 37. Let G be graph with n vertices and minimum degree $\delta(G)$ and maximum degree $\Delta(G)$. Then

$$\frac{n}{2}\sqrt{\frac{\delta(G)}{\Delta(G)}}\leq P(G)\leq\frac{n}{2} \qquad (4)$$

Proof: (4) follows from inequality (1) and inequality (2).

We use the following result to obtain our next result.

Theorem 38 [22]. For any (n, m)-connected graph G with $\delta(G) \geq 2$ and $n \geq 3$ vertices,

$$\frac{n\sqrt{2}}{\sqrt{(n-1)(n-2)}}\leq PB(G)\leq n.$$

Further equality holds in lower bound if and only if $G = C_3$ and equality holds in upper bound if and only if $G = C_n$, $n \geq 3$.

Theorem 39. For any (n, m)-connected graph G with $\delta(G) \geq 2$ and $n \geq 3$ vertices,

$$\frac{n}{2}\sqrt{\frac{\delta(G)}{\Delta(G)}}+\frac{n\sqrt{2}}{\sqrt{(n-1)(n-2)}}+PEM(G)\leq PMB(G)\leq\frac{3}{2}n+PEM(G).$$

Proof: From Theorem 20, we have

$$PMB(G)=P(G)+PB(G)+PEM(G).$$

Using inequality (4), we obtain

$$\frac{n}{2}\sqrt{\frac{\delta(G)}{\Delta(G)}}+PB(G)+PEM(G)\leq PMB(G)\leq\frac{n}{2}+PB(G)+PEM(G).$$

Using Theorem 38, we obtain

$$\frac{n}{2}\sqrt{\frac{\delta(G)}{\Delta(G)}}+\frac{n\sqrt{2}}{\sqrt{(n-1)(n-2)}}+PEM(G)\leq PMB(G)\leq\frac{3}{2}n+PEM(G).$$

Theorem 40 [22. For any (n, m)-connected with $\delta(G) \geq 2$ and $n \geq 3$ vertices,

$$\sqrt{\frac{2\delta(G)}{\Delta(G)-1}}\delta(G)M_2^*(G)\leq PB(G)\leq\sqrt{\frac{2\Delta(G)}{\delta(G)-1}}\Delta(G)M_2^*(G).$$

Theorem 41. For any (n, m)-connected with $\delta(G) \geq 2$ and $n \geq 3$ vertices,

$$\frac{n}{2}\sqrt{\frac{\delta(G)}{\Delta(G)}}+\sqrt{\frac{2\delta(G)}{\Delta(G)-1}}\delta(G)M_2^*+PEM(G)\leq PMB(G)\leq\frac{n}{2}+\sqrt{\frac{2\Delta(G)}{\delta(G)-1}}\Delta(G)M_2^*(G)+PEM(G).$$

Proof: From Theorem 20, we have

$$PMB(G)=P(G)+PB(G)+PEM(G).$$

Using inequality (4), we obtain

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$$\frac{n}{2}\sqrt{\frac{\delta(G)}{\Delta(G)}}+PB(G)+PEM(G)\leq PMB(G)\leq\frac{n}{2}+PB(G)+PEM(G).$$

Using Theorem 40, we get

$$\frac{n}{2}\sqrt{\frac{\delta(G)}{\Delta(G)}}+\delta(G)M_2^*(G)+PEM(G)\leq PMB(G)\leq\frac{n}{2}+\sqrt{\frac{2\Delta(G)}{\delta(G)-1}}\Delta(G)M_2^*(G)+PEM(G).$$

Theorem 42[22]. For any (n, m)-connected graph G with $\delta(G)\geq 2$ and $n\geq 3$ vertices,

$$\frac{m}{\Delta(G)}\sqrt{\frac{2\delta(G)}{\Delta(G)-1}}\leq PB(G)\leq\frac{m}{\delta(G)}\sqrt{\frac{2\Delta(G)}{\delta(G)-1}}.$$

Equality in both lower and upper bounds will hold if and only if G is regular.

Theorem 43. For any (n, m)-connected with $\delta(G)\geq 2$ and $n\geq 3$ vertices,

$$\frac{n}{2}\sqrt{\frac{\delta(G)}{\Delta(G)}}+\frac{m}{\Delta(G)}\sqrt{\frac{2\delta(G)}{\Delta(G)-1}}+PEM(G)\leq PMB(G)\leq\frac{n}{2}+\frac{m}{\delta(G)}\sqrt{\frac{2\Delta(G)}{\delta(G)-1}}+PEM(G).$$

Proof: From Theorem 20, we have

$$PMB(G)=P(G)+PB(G)+PEM(G).$$

Using inequality (4), we get

$$\frac{n}{2}\sqrt{\frac{\delta(G)}{\Delta(G)}}+PB(G)+PEM(G)\leq PMB(G)\leq\frac{n}{2}+PB(G)+PEM(G).$$

Using Theorem 42, we obtain

$$\frac{n}{2}\sqrt{\frac{\delta(G)}{\Delta(G)}}+\frac{m}{\Delta(G)}\sqrt{\frac{2\delta(G)}{\Delta(G)-1}}+PEM(G)\leq PMB(G)\leq\frac{n}{2}+\frac{m}{\delta(G)}\sqrt{\frac{2\Delta(G)}{\delta(G)-1}}+PEM(G).$$

We use the following results to establish our next result.

Theorem 44 [33]. Let G be a simple connected graph with n vertices and m edges. Let p, $\Delta(G)$, $\delta_1(G)$ denote the number of pendant vertices, maximum vertex degree and minimum nonpendant vertex degree of G respectively. Then

$$\frac{p}{\sqrt{\Delta(G)}}+\frac{2\sqrt{\delta_1(G)\Delta(G)(m-p)}}{\delta_1(G)+\Delta(G)}\sqrt{M_2^*(G)-\frac{p}{\Delta(G)}}\leq P(G)\leq$$

$$\frac{p}{\sqrt{\delta_1(G)}}+\sqrt{(m-p)\left(M_2^*(G)-\frac{p}{\delta_1(G)}\right)}$$

Theorem 45 [22]. For any (n, m)-connected graph G with p pendant vertices and minimum nonpendant vertex degree $\delta_1(G)$,

$$\frac{p\left(1+\sqrt{\Delta(G)}\right)+(m-p)\sqrt{2}}{\sqrt{\Delta(G)\Delta(G)-1}}\leq PB(G)\leq\frac{p\left(1+\delta_1(G)\right)+(m-p)\sqrt{2}}{\sqrt{\delta_1(G)+\left(\delta_1(G)-1\right)}}.$$

Theorem46. For any (n, m)-connected graph G with p pendant vertices, maximum vertex degree $\Delta(G)$, minimum nonpendant vertex degree $\delta_1(G)$,

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$$\frac{p}{\sqrt{\Delta(G)}}+\frac{2\sqrt{\delta_1(G)\Delta(G)(m-p)}}{\delta_1(G)+\Delta(G)}\sqrt{M_2^*(G)-\frac{p}{\Delta(G)}}+\frac{p\left(1+\sqrt{\Delta(G)}\right)+(m-p)\sqrt{2}}{\sqrt{\Delta(G)\Delta(G)-1}}+PEM(G)$$

$$\leq PMB(G)\leq\frac{p}{\sqrt{\Delta(G)}}+\sqrt{(m-p)\left(M_2^*(G)-\frac{p}{\Delta(G)}\right)}+\frac{p(1+\delta_1(G))+(m-p)\sqrt{2}}{\sqrt{\delta_1(G)+(\delta_1(G)-1)}}+PEM(G)$$

Proof: From Theorem 20, we have

$$PMB(G)=P(G)+PB(G)+PEM(G).$$

Then from Theorems 44 and 45, we get the desired result.

4. CONCLUSION

In this study, we have introduced the sum connectivity Zagreb-*K*-Banhatti index and product Zagreb-*K*-Banhatti index of a graph. We have established lower and upper bounds for these two connectivity Zagreb-K-Banhatti indices of a connected graph in terms of Zagreb indices.

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